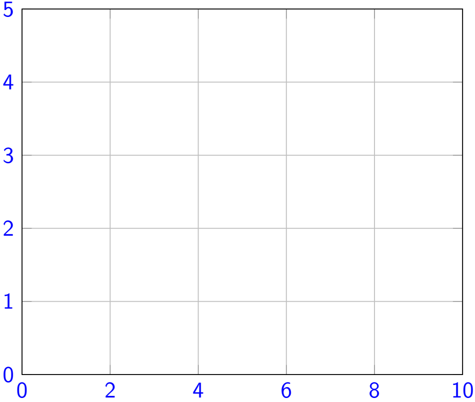
Chapter 4.7: Newton's Method

Newton's Method - Idea

Goal:
Get approximation of roots.
That is, solve $f(x) = 0$.

Idea:
It is easy to find a root of a line.
If we have a reasonable guess, we can improve it by approximating f by a tangent line.

Use: Recall, for min/max of f, we need to solve $f'(x) = 0$.



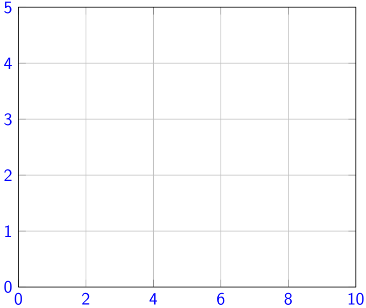
Newton's Method - Formula

Outline of the method:
Start with an initial guess and keep improving it.
Good initial guess is important!

- Start with initial guess x_0.
- Repeatedly apply the following formula to get (hopefully) better approximations.

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- Stop when approximation is sufficient.
  Or after some number of steps, or if it starts exploding.



$$0 = f(a) + f'(a)(b-a)$$
$$-f(a) = f'(a)(b-a)$$
$$\frac{-f(a)}{f'(a)} = b - a$$
$$\boxed{b = a - \frac{f(a)}{f'(a)}}$$

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $\sqrt{2}$

Set $f(x) = x^2 - 2$. Then $f'(x) = 2x$ and so Newton's method becomes

$$x_{n+1} = x_n - \frac{(x_n)^2 - 2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n}$$

Let's start with $x_1 = 1$, then

$$x_2 = \frac{1}{2} + 1 = \frac{3}{2} = 1.5$$
$$x_3 = \frac{3/2}{2} + \frac{1}{3/2} = \frac{17}{12} \approx 1.416$$
$$x_4 = \frac{17/12}{2} + \frac{1}{17/12} = \frac{577}{408} \approx 1.4142$$

As we may know, $\sqrt{2} \approx 1.4142$ and so we have that $577/408$ is a pretty good approximation of $\sqrt{2}$.

This method will not find the exact root, only (good) approximation.

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $2x = \cos(x)$ starting with $x_1 = 0$.

Indeed, let $f(x) = 2x - \cos(x)$, then $f'(x) = 2 + \sin(x)$ and so Newton's method is

$$x_{n+1} = x_n - \frac{2x_n - \cos(x_n)}{2 + \sin(x_n)}$$

Starting with $x_1 = 0$ gives

$$x_2 = 0 - \frac{2(0) - \cos(0)}{2 + \sin(0)} = \frac{1}{2} = .5$$

$$x_3 = \frac{1}{2} - \frac{2(1/2) - \cos(1/2)}{2 + \sin(1/2)} \approx 0.4506$$

$$x_4 \approx 0.4506 - \frac{2(0.4506) - \cos(0.4506)}{2 + \sin(0.4506)} \approx 0.4501$$

Correct value is around $0.450184\ldots$.

Example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Example: Approximate $x^3 - x = 1$

starting with $x = 1$.

Set $f(x) = x^3 - x - 1$. Then $f'(x) = 3x^2 - 1$.

It looks like x_{n+1} does not have a nice formula so we don't bother. and we make a table instead:

x_n	$f(x_n)$	$f'(x_2)$	x_{n+1}
1	-1	-2	$1 - \frac{-1}{2} = \frac{3}{2}$
$\frac{3}{2}$	0.875	5.75	≈ 1.3478
1.3478	$.10056$	4.4496	$1.32519 ldots$

The real solution is 1.3247.

starting with $x = 0$.

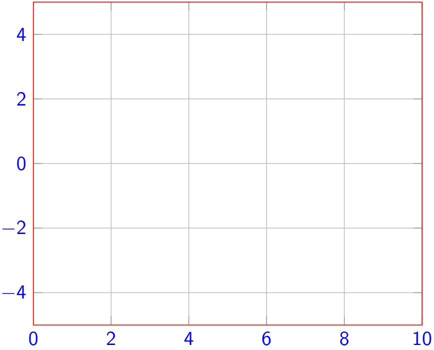
x_n	$f(x_n)$	$f'(x_2)$	x_{n+1}
0	-1	-1	$1 - \frac{-1}{-1} = -1$
-1	-1	2	$-1 - \frac{-1}{2} = 0.5$
-0.5	$-.625$	-0.25	-3

Does not look good at all... Not converging towards 1.3247...

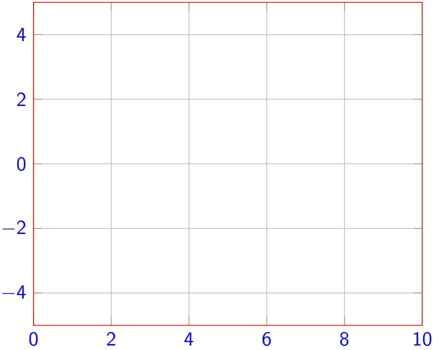
Fails

Newtons method may fail in many ways, such as division by zero, converging to a different root, not converging at all or even diverging. Initial guess is important!

Converging to a different root



Not converging at all



I promise that this is very rare and Newton's method is great!

Failing example $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Try Newton's method for

$$f(x) = \sqrt{|x|}$$

with initial guess $x_1 = 1$. Note that

$$f'(x) = \frac{1}{2\sqrt{|x|}} \cdot \frac{x}{|x|} = \frac{x}{2|x|\sqrt{|x|}}.$$

Newton's method becomes

$$x_{n+1} = x_n - \frac{\sqrt{|x_n|}}{\frac{x_n}{2|x_n|\sqrt{|x_n|}}} = x_n - \frac{2|x_n|^2}{x_n} = x_n - \frac{2x_n^2}{x_n} = x_n - \frac{2}{x_n}$$

Starting with $x_1 = 1$ then yields

$$x_2 = 1 - \frac{2}{1} = -1$$

$$x_3 = -1 - \frac{2}{-1} = 1$$

$$x_4 = 1 - \frac{2}{1} = -1$$